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Isotropic A-branes and the stability condition

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Abstract: The existence of a new kind of branes for the open topological A-model is argued by using the generalized complex geometry of Hitchin and the SYZ picture of mirror symmetry. Mirror symmetry suggests to consider a bi-vector in the normal direction of the brane and a new definition of generalized complex submanifold. Using this definition, it is shown that there exist generalized complex submanifolds which are isotropic in a symplectic manifold. For certain target space manifolds this leads to isotropic A-branes, which should be considered in addition to lagrangian and coisotropic A-branes. The Fukaya category should be enlarged with such branes, which might have interesting consequences for the homological mirror symmetry of Kontsevich. The stability condition for isotropic A-branes is studied using the worldsheet approach.

Keywords: Topological Field Theories, D-branes, Differential and Algebraic Geometry.

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Contents

1. Introduction

Since the seminal paper of Witten [1] it is known that there are topological A-branes which are lagrangian submanifolds. This was part of the boundary conditions imposed to the topological A-model formulated on a Riemann surface with boundaries. The quantum version of the open topological model was obtained adding a Wilson loop to the path integral of the closed topological model. Requiring the Wilson loop to be BRST invariant, it was found that lagrangian submanifolds can carry only a flat vector bundle.

The possibility of a non-flat connection for a D-brane was considered in [2, 3]. The algebraic condition that the curvature of a line bundle over the worldvolume of the brane has to satisfy in order that the D-brane is a topological A-brane has been worked out in [4]. These topological branes are non-lagrangian branes of higher dimension than lagrangian ones and they have been called coisotropic branes using the language of symplectic geometry. One of the main motivations to consider such branes was in the context of homological mirror symmetry. It was argued that Kontsevich's conjecture [5] can be true only if the Fukaya category is enlarged with coisotropic A-branes.

When the target space manifold has a generalized complex geometry [6, 7], B and A-twisted topological models have been defined in [8, 9]. The paper [8] also treats the case of topological branes showing that their geometry is naturally described in terms of generalized complex submanifolds. In particular, coisotropic A-branes are generalized complex submanifolds of a symplectic manifold [7].

In a recent paper [10] mirror symmetry in the Strominger-Yau-Zaslow (SYZ) picture of T-duality [11] is studied in the context of topological models with generalized complex geometry. Under the explicitly constructed mirror map topological A and B-branes are mirror pairs. Furthermore, it was shown that the field strength on the world volume of the brane is mapped to a bi-vector in the normal direction of the mirror brane. This leads to a definition of a generalized tangent bundle when there is a bi-vector in the normal direction of a generalized submanifold.

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In this paper we use the above insight from mirror symmetry to analyze generalized topological A-branes which are equipped with a bi-vector in the normal direction of the brane. In this case the definition of a generalized complex submanifold has to be refined with respect to the case that there is a two-form on the submanifold [7]. Recall that the two-form on the submanifold is necessary to have covariance under B-field transformations. This is because a B-field transformation acts as a shear transformation on the cotangent bundle T^*. Since a β-field transformation acts as a shear transformation on the tangent bundle T, the bi-vector turns out to give covariance under β-field transformations. As a natural consequence one finds generalized complex submanifolds which are isotropic in a symplectic manifold. The target space of the generalized topological A-model with complex structures for left and right movers identified is in particular a symplectic manifold and therefore one finds isotropic A-branes. Denoting with n the complex dimension of the target space N, this topological branes are of real dimension $n-2k$, which has to be compared with the real dimension $n+2k$ of coisotropic A-branes. Isotropic branes are non-trivial cycles if the Betti numbers $b_{n-2k}(N)$ are non zero. By Poincaré duality $b_{n-2k}(N) = b_{n+2k}(N)$ and therefore isotropic and coisotropic branes appear in pairs but with different differential structures on them.

The stability condition for isotropic A-branes is also worked out following the worldsheet approach applied already to the case of coisotropic A-branes [12]. This is an important issue because it leads to supersymmetric cycles and to the existence of BPS states.

The paper is organized as follows. In section (2) we take a D2 topological B-brane wrapping a two dimensional torus fiber T^2 in the target space T^6, which is a trivial T^3 torus fibration. We consider this case to construct explicitly the generalized tangent bundle on the mirror side. In section (3) we use the definition of a generalized complex submanifold equipped with a bi-vector in the normal direction to show the existence of isotropic A-branes. In section (4) we work out the condition that a certain form of the target space has to satisfy in order that isotropic A-branes are stable. In the last section of the paper we give the conclusion and outlook.

2. D2-brane on the fiber and the mirror brane

Topological A and B-branes are submanifolds M of the target space N of the twisted $\mathcal{N} = (2,2)$ supersymmetric non linear sigma model in two dimensions. The submanifold M is a topological A or B-brane if the U(1) R-currents $j_\pm = \omega_\pm(\psi_\pm,\, \psi_\pm)$ match on the boundary of the Riemann surface as explained below [8]. Note that we allow for the most general target space geometry, which is a bi-hermitean geometry described in terms of two different complex structures I_+ and I_- for right and left movers.[1] This geometry was first discovered in [13] and later included in the framework of Hitchin's generalized complex geometry [6] by Gualtieri [7]. The conditions $j_+ + j_- = 0$ and $j_+ - j_- = 0$ lead to topological A and B-branes respectively and preserve $\mathcal{N} = 2$ worldsheet supersymmetry on the boundary. One has also to require boundary conditions for the fermions arising from

[1]Bi-hermitean means that the riemannian metric $G \in \odot^2 T_N^*$ is hermitean with respect to I_+ and I_-. There are also two symplectic structures $\omega_\pm = GI_\pm$.

$\mathcal{N}=1$ worldsheet supersymmetry, which are described in terms of a gluing matrix R and take the form $\psi_- = R\psi_+$. The full set of boundary conditions preserving $\mathcal{N}=1$ worldsheet supersymmetry have been studied in [14, 15]. If there exists a two form $F \in \Lambda^2 T^*_M$ and $I_+ = I_-$, the matrix R satisfies the equation $\pi^t(G-F)\pi = \pi^t(G+F)\pi R$, where π is the projector on the Neumann directions of the brane. Using the form of the projector π in [15], one finds (see e.g. [4]) $R^{-1} = (-id_{N_M}) \oplus (g-F)^{-1}(g+F)$ on $T_N|_M = N_M \oplus T_M$, where N_M is the normal bundle to M and g is the restriction of the full metric G to T_M.

When $I_+ \neq I_-$ the geometry of topological branes is naturally described using the language of generalized complex geometry [8]. The brane geometry was then described in terms of a gluing matrix $\mathcal{R} : T_N \oplus T^*_N \to T_N \oplus T^*_N$ in [16]. The gluing condition becomes $\Psi = \mathcal{R}\Psi$ with

$$\mathcal{R} = \begin{pmatrix} 1 & \\ F & 1 \end{pmatrix} \begin{pmatrix} r & \\ & -r^t \end{pmatrix} \begin{pmatrix} 1 & \\ -F & 1 \end{pmatrix} = \begin{pmatrix} r & \\ Fr + r^t F & -r^t \end{pmatrix} . \tag{2.1}$$

In the last equation $r = \pi - Q$, where Q is the projector on the Dirichlet directions of the brane and

$$\Psi := \begin{pmatrix} \psi \\ \rho \end{pmatrix}, \qquad \psi := \frac{1}{2}(\psi_+ + \psi_-) \in T_N\,, \qquad \rho := \frac{1}{2}G(\psi_+ - \psi_-) \in T^*_N\,. \tag{2.2}$$

Using the projector $\mathcal{R}_+ = (1+\mathcal{R})/2$ and defining

$$\tau_+ := \left\{ X + \xi \in (T_N \oplus T^*_N)|_M : \mathcal{R}_+(X+\xi) = X + \xi \right\}, \tag{2.3}$$

one finds

$$\tau_+ \equiv \tau^F_M = \left\{ X + \xi \in T_M \oplus T^*_N|_M : \xi|_M = X \lrcorner F \right\}. \tag{2.4}$$

τ^F_M is the generalized tangent bundle of Gualtieri [7]. By the conditions $j_+ \pm j_- = 0$ written in terms of the $T_N \oplus T^*_N$ bundle, topological A (B) branes are described in terms of the generalized Kähler structure $\mathcal{J}_2$ ($\mathcal{J}_1$) defined in [7]. Since the matrix $\mathcal{R}$ commutes with $\mathcal{J}_2$ ($\mathcal{J}_1$) for topological A (B) branes, topological branes are generalized complex submanifolds $(M, F) \subset (N, \mathcal{J}_{2/1})$ according to the definition given in [7].

The rest of the section is dedicated to show that by applying the SYZ picture of mirror symmetry to topological branes, one obtains a new kind of generalized tangent bundle. This definition was already given in [10]. In that paper, the target space is a six dimensional torus considered as a trivial T^3 torus fibration. Thus, $N = \mathcal{B} \oplus \mathcal{F} := T^3 \oplus T^3$, where $\mathcal{B}$ and $\mathcal{F}$ are the base and fiber spaces. The case of a D3 topological A-brane wrapping $\mathcal{F}$ was considered.[2] Then, under the explicitly constructed mirror map the D3-brane is mapped to a D0 topological B-brane and the field strength F on the worldvolume of the D3-brane is mapped to a bi-vector $\tilde{\beta} = F^{-1}$. This bi-vector lives on the fiber, which is now the normal direction of the D0-brane.

Here, we want to consider the case of a D2 topological B-brane to make the appearance of the new generalized tangent bundle explicit. Consider the torus fiber as $\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2 :=$

[2] We use a notation where the worldvolume of a Dp-brane is p-dimensional.

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$T^2 \oplus T$. Then, wrap the two dimensional torus T^2 with a D2 topological B-brane having a field strength on the worldvolume, i.e. $F \in \Lambda^2 T^*_{\mathcal{F}_1}$. The gluing matrix $\mathcal{R}$ in (2.1) becomes

$$\mathcal{R} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2F & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.5}$$

which commutes with $\mathcal{J}_1$. The mirror map

$$\mathcal{M} : T_{\mathcal{B}} \oplus T_{\mathcal{F}_1} \oplus T_{\mathcal{F}_2} \oplus T^*_{\mathcal{B}} \oplus T^*_{\mathcal{F}_1} \oplus T^*_{\mathcal{F}_2} \to T_{\mathcal{B}} \oplus T^*_{\mathcal{F}_1} \oplus T^*_{\mathcal{F}_2} \oplus T^*_{\mathcal{B}} \oplus T_{\mathcal{F}_1} \oplus T_{\mathcal{F}_2} \tag{2.6}$$

is realized explicitly as

$$\mathcal{M} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \tag{2.7}$$

under which the gluing matrix gets mapped to

$$\hat{\mathcal{R}} = \mathcal{M}\mathcal{R}\mathcal{M}^{-1} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 2\tilde{\beta} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \tag{2.8}$$

with $\tilde{\beta} = F^{-1} \in \Lambda^2 T_{\mathcal{F}_1}$. The above mirror map also exchanges $\mathcal{J}_1$ with $\mathcal{J}_2$ [10] and therefore $\hat{\mathcal{R}}$ commutes with $\mathcal{J}_2$. We see that Neumann and Dirichlet boundary conditions are interchanged so that the D2 topological B-brane is mapped to a D1 topological A-brane wrapping $\mathcal{F}_2$. The field strength F is mapped to a bi-vector $\tilde{\beta}$ living on $\mathcal{F}_1$, which is now the normal direction of the D1-brane. Computing the projector $\hat{\mathcal{R}}_+ := (1 + \hat{\mathcal{R}})/2$, one obtains

$$\hat{\mathcal{R}}_+ \begin{pmatrix} x_B \\ x_1 \\ x_2 \\ \xi_B \\ \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \tilde{\beta}(\xi_1) \\ x_2 \\ \xi_B \\ \xi_1 \\ 0 \end{pmatrix}, \tag{2.9}$$

where $x_1 \in T_{\mathcal{F}_1}$ and $x_2 \in T_{\mathcal{F}_2}$ (the ξ's are the dual coordinates). Thus, one sees that defining

$$\rho^{\tilde{\beta}}_{\mathcal{F}_2} := \left\{ X + \xi \in (T_N \oplus T^*_N)|_{\mathcal{F}_2} : \hat{\mathcal{R}}_+(X + \xi) = X + \xi \right\}, \tag{2.10}$$

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we have

$$\rho^{\tilde{\beta}}_{\mathcal{F}_2} = \left\{X + \xi \in T_{\mathcal{F}}|_{\mathcal{F}_2} \oplus (T^*_{\mathcal{B}} \oplus T^*_{\mathcal{F}_1})|_{\mathcal{F}_2} : x_1 = \tilde{\beta}(\xi_1)\right\}, \tag{2.11}$$

which is stable under $\mathcal{J}_2$.

The study of the torus indicates that the D2 topological B-brane $(M, F) \subset (N, \mathcal{J}_1)$ — τ^F_M is stable under $\mathcal{J}_1$ — is mapped under the mirror map to a D1 topological A-brane $(\hat{M}, \tilde{\beta}) \subset (\hat{N}, \mathcal{J}_2)$ with $\tilde{\beta} = F^{-1} \in \Lambda^2 N_{\hat{M}}$ and[3]

$$\rho^{\tilde{\beta}}_{\hat{M}} := \left\{X + \xi \in T_{\hat{N}}|_{\hat{M}} \oplus N^*_{\hat{M}} \,:\, X|_{N^*_{\hat{M}}} = \tilde{\beta}(\xi)\right\} \tag{2.12}$$

stable under $\mathcal{J}_2$. This lead us to the following definition. The generalized submanifold $(M, \tilde{\beta}) \subset (N, \mathcal{J})$, with $\tilde{\beta} \in \Lambda^2 N_M$, is a generalized complex submanifold if $\rho^{\tilde{\beta}}_M$ is stable under $\mathcal{J}$. In the next section we will see that with this notion of a generalized complex submanifold, in the symplectic case one finds isotropic A-branes.

3. Isotropic A-branes

In this section we want to use generalized complex geometry to show that there are generalized complex submanifolds of a symplectic manifold which are isotropic. To this end we need the definition of a generalized complex submanifold given in the last section, which we repeat here for clarity.

Definition 1. *The generalized tangent bundle of a generalized submanifold* $(M, \tilde{\beta})$*, with* $\tilde{\beta} \in \Lambda^2 N_M$*, is defined by*

$$\rho^{\tilde{\beta}}_M := \left\{X + \xi \in T_N|_M \oplus N^*_M \,:\, X|_{N^*_M} = \tilde{\beta}(\xi)\right\}.$$

Definition 2. *The generalized submanifold* $(M, \tilde{\beta}) \subset (N, \mathcal{J})$*, with* $\tilde{\beta} \in \Lambda^2 N_M$*, is a generalized complex submanifold if* $\rho^{\tilde{\beta}}_M$ *is stable under* $\mathcal{J}$*.*

When M is equipped with a 2-form $F \in \Lambda^2 T^*_M$, the corresponding definitions are given in [7]. In this case the generalized tangent bundle is given by (2.4), which transforms naturally under B-field transformations. In this paper we consider M equipped with a bi-vector $\tilde{\beta} \in \Lambda^2 N_M$ and therefore the generalized tangent bundle is $\rho^{\tilde{\beta}}_M$, which transforms naturally under β-field transformations.

From now on the conormal bundle N^*_M is denoted with the annihilator $\mathrm{Ann}\, T_M$. For the concepts of symplectic geometry used in this section we refer the reader to [17]. Let N be a symplectic manifold endowed with the symplectic structure ω and $(M, \tilde{\beta}) \subset (N, \mathcal{J}_\omega)$ its generalized complex submanifold with respect to the generalized complex structure

$$\mathcal{J}_\omega = \begin{pmatrix} & -\omega^{-1} \\ \omega & \end{pmatrix}. \tag{3.1}$$

[3] N_M is the normal bundle of M in N defined as the quotient $T_N|_M / T_M$. Note that the conormal bundle N^*_M is the annihilator $\mathrm{Ann}\, T_M$.

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Note that for generalized topological A-branes we have to consider $\mathcal{J}_2$, which reduces to the symplectic case $\mathcal{J}_\omega$ for topological A-branes (in Witten's sense of [1] where $I_+ = I_-$). Choosing $\beta|_{\mathrm{Ann}\, T_M} = \tilde{\beta}$, the bundle $\rho_M^{\tilde{\beta}}$ is stable under $\mathcal{J}$ iff

$$e^{-\beta}\rho_M^{\tilde{\beta}} = \rho_M^0 = T_M \oplus \mathrm{Ann}\, T_M \tag{3.2}$$

is stable under $e^{-\beta}\mathcal{J}e^{\beta}$. Here,

$$e^{\beta} = \begin{pmatrix} 1 & \beta \\ & 1 \end{pmatrix} \tag{3.3}$$

is the β-transform of Gualtieri [7]. Requiring the stability of (3.2) under

$$e^{-\beta}\mathcal{J}_\omega e^{\beta} = \begin{pmatrix} -\beta\omega & -\omega^{-1} - \beta\omega\beta \\ \omega & \omega\beta \end{pmatrix}, \tag{3.4}$$

one obtains the following conditions:

1. $\omega : T_M \to \mathrm{Ann}T_M$, i.e. M is an isotropic manifold,
2. $\omega\beta : \mathrm{Ann}T_M \to \mathrm{Ann}T_M$,
3. $\beta\omega : T_M \to T_M$,
4. $\omega^{-1} + \beta\omega\beta : \mathrm{Ann}T_M \to T_M$.

Let us define the symplectic complement of T_M in $T_N|_M$:

$$T_M^\omega := \left\{ v \in T_N|_M : \omega(v,w) = 0 \ \ \forall w \in T_M \right\}. \tag{3.5}$$

The annihilators $\mathrm{Ann}\, T_M$ and $\mathrm{Ann}\, T_M^\omega$ are defined as follows:

$$\begin{aligned} \mathrm{Ann}\, T_M &:= \left\{ \eta \in T_N^*|_M : \eta(v) = 0 \ \ \forall v \in T_M \right\}, \\ \mathrm{Ann}\, T_M^\omega &:= \left\{ \eta \in T_N^*|_M : \eta(v) = 0 \ \ \forall v \in T_M^\omega \right\}. \end{aligned} \tag{3.6}$$

Under the isomorphism

$$\begin{aligned} (T_N)_x &\to (T_N^*)_x \\ v &\mapsto v \lrcorner\, \omega(x). \end{aligned} \tag{3.7}$$

the bundle T_M^ω is identified with the annihilator $\mathrm{Ann}\, T_M$. This means that $\forall v_\eta \in (T_M^\omega)_x$ with $x \in M$, there exists one $\eta \in \mathrm{Ann}(T_M)_x$ such that $\eta = v_\eta \lrcorner\, \omega(x)$. One also finds that:

- If M is isotropic in N, $T_M \subset T_M^\omega$ which implies that $\mathrm{Ann}\, T_M^\omega \subset \mathrm{Ann}\, T_M$.
- If M is coisotropic in N, $T_M^\omega \subset T_M$ which implies that $\mathrm{Ann}\, T_M \subset \mathrm{Ann}\, T_M^\omega$.
- If M is lagrangian in N, $T_M = T_M^\omega$ which implies that $\mathrm{Ann}\, T_M = \mathrm{Ann}\, T_M^\omega$.

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We want to show that the first and second conditions above lead to $\tilde{\beta}|_{\mathrm{Ann}\,T^\omega_M} = 0$. The second one tells us that $\omega\tilde{\beta}(\xi) = \eta$ with $\eta \in \mathrm{Ann}T_M$, $\forall\xi \in \mathrm{Ann}T_M$. The nondegeneracy of the symplectic structure in N yields $\tilde{\beta}(\xi,\sigma) = \omega^{-1}(\eta,\sigma)$ $\forall\xi,\sigma \in \mathrm{Ann}T_M$ and $\eta \in \mathrm{Ann}\,T_M$. By means of the isomorphism (3.7), the last equation can be rewritten as $\tilde{\beta}(\xi,\sigma) = \sigma(v_\eta)$ $\forall\xi,\sigma \in \mathrm{Ann}T_M$ and $v_\eta \in T^\omega_M$. Using the isotropic property $\mathrm{Ann}\,T^\omega_M \subset \mathrm{Ann}\,T_M$ and the definition (3.6) of $\mathrm{Ann}\,T^\omega_M$, one obtains $\tilde{\beta}(\xi,\sigma) = 0$ $\forall\xi \in \mathrm{Ann}\,T_M$ and $\forall\sigma \in \mathrm{Ann}\,T^\omega_M$. We can use again the property $\mathrm{Ann}\,T^\omega_M \subset \mathrm{Ann}\,T_M$ and get $\tilde{\beta}(\xi,\sigma) = 0$ $\forall\xi,\sigma \in \mathrm{Ann}\,T^\omega_M$, which just means that $\tilde{\beta}|_{\mathrm{Ann}\,T^\omega_M} = 0$. Note that this condition implies that $\tilde{\beta} = 0$ if M is a lagrangian submanifold.

Defining $E := T^\omega_M/T_M$, the consequence of $\tilde{\beta}|_{\mathrm{Ann}\,T^\omega_M} = 0$ is that $\omega|_E\tilde{\beta}$ becomes an almost complex structure on E^*. In fact, given the natural isomorphism $\mathrm{Ann}\,T_M \simeq \mathrm{Ann}\,T^\omega_M \oplus E^*$, $\tilde{\beta} \in \Lambda^2 N_M$ descends to a bi-vector $\tilde{\beta} \in \Lambda^2 E$. Furthermore, using the fourth condition above, one finds $\omega|_E\tilde{\beta} : E^* \to E^*$ and $(\omega|_E\tilde{\beta})^2 = -1$, i.e. $\omega|_E\tilde{\beta}$ is an almost complex structure on E^*.

One can show that $\tilde{\beta} \pm i\omega^{-1}|_E$ are respectively nondegenerate $(2,0)$ and $(0,2)$ vectors with respect to the almost complex structure on E^*. Therefore, $E^{(1,0)}_x$ is of even complex dimension and $\dim|_{\mathbb{R}}E_x = \dim|_{\mathbb{C}}(E\otimes\mathbb{C})_x = 4k$. By the dimension theorem for the symplectic complement, we have $\dim|_{\mathbb{R}}(T_N)_x = \dim|_{\mathbb{R}}(T_M)_x + \dim|_{\mathbb{R}}(T^\omega_M)_x$ and in addition $\dim|_{\mathbb{R}}(T^\omega_M)_x = \dim|_{\mathbb{R}}E_x + \dim|_{\mathbb{R}}(T_M)_x$. Therefore, $\dim|_{\mathbb{R}}M = n - 2k$ with $n = (\dim|_{\mathbb{R}}N)/2$. Note that $k = 0, 1, \dots n/2$ for n even and $k = 0, 1, \dots (n-1)/2$ for n odd. For $k = 0$, $\dim|_{\mathbb{R}}M = n$ and M is a lagrangian submanifold.

The results above show the existence of generalized complex submanifolds M of real dimension $n-2k$ which are isotropic in the symplectic manifold N of real dimension $2n$. We recall that the topological B-model is well defined at the quantum level if N is a Calabi-Yau manifold but for the topological A-model N can also be a Kähler manifold. The conclusions above imply that there could be isotropic A-branes in addition to the lagrangian A-branes of Witten [1] and coisotropic A-branes of Kapustin and Orlov [4]. Since we want to see under which conditions there are isotropic A-branes besides lagrangian ones, we shall take N to be a Kähler or Calabi-Yau manifold[4] and $k \neq 0$. If all Betti numbers $b_{n-2k}(N)$ are equal to zero, there are no isotropic A-branes because they are homologically trivial. However, one has to expect their appearance whenever $b_{n-2k}(N) \neq 0$. If M' is a coisotropic A-brane, $\dim|_{\mathbb{R}}M' = n + 2k$. By Poincaré duality, $b_{n-2k}(N) = b_{n+2k}(N)$ and the same considerations about homological triviality for coisotropic branes [4] apply to isotropic branes. For instance, we know that $b_1(N) = 0$ if N is a compact riemannian manifold with holonomy group $\mathrm{SU}(n)$ [18]. Therefore, there are no one dimensional isotropic A-branes if N is a compact odd dimensional Calabi-Yau manifold with holonomy $\mathrm{SU}(n)$. This means that the only compact Calabi-Yau 3-folds N which allow isotropic A-branes are $N = T^6$ and $N = T^2 \times K3$, which have holonomy group smaller than $\mathrm{SU}(3)$. There are also no isotropic A-branes for complete intersection Calabi-Yau manifolds of odd dimensions in a projective space because all odd Betti numbers, with the exception of $b_n(N)$ corresponding

[4]For a Calabi-Yau n-fold we denote, perhaps with a slight abuse of terminology, a Kähler manifold with holonomy group smaller or equal to $\mathrm{SU}(n)$.

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to lagrangian submanifolds, are zero. The same consideration does not apply when the dimension is even because the even Betti numbers are different from zero and therefore there are isotropic A-branes.

4. Stability condition for isotropic A-branes

The aim of this section is to compute the stability condition for isotropic A-branes following the worldsheet approach used in [12] to find stability for coistropic A-branes. In section (2), when M is equipped with a 2-form $F \in \Lambda^2 T^*_M$, the gluing matrix $\mathcal{R} : T_N \oplus T^*_N \to T_N \oplus T^*_N$ was constructed from the boundary conditions written in terms of the gluing matrix $R : T_N \to T_N$ satisfying the equation $\pi^t(G - F)\pi = \pi^t(G + F)\pi R$. Here, the manifold M is equipped with a bi-vector $\tilde{\beta} \in \Lambda^2 N_M$ and we follow the opposite approach. We first construct the gluing matrix $\mathcal{R}$ which leads to a generalized tangent bundle τ_+ defined in (2.3) equal to the generalized tangent bundle $\rho^{\tilde{\beta}}_M$ defined in the previous section. Then, we find the condition the gluing matrix R has to satisfy by projecting on the T_N bundle.

In the paper [14] the $\mathcal{N} = 1$ boundary conditions were studied when $F = 0$. In a following paper [15] the case with $F \neq 0$ was considered. In the latter paper the Dirichlet boundary conditions are unmodified with respect to the case $F = 0$. It was required that $QR = RQ = -Q$, where Q is the projector on the Dirichlet directions, i.e.

$$Q = \begin{pmatrix} & \\ & \delta^i{}_j \end{pmatrix}, \qquad R = \begin{pmatrix} R^m{}_n & \\ & -\delta^i{}_j \end{pmatrix}. \tag{4.1}$$

The indices i, j and m, n label the Dirichlet and Neumann directions respectively. Then, the projector on the Neumann directions is defined by $\pi = 1 - Q$. In our case with $\tilde{\beta} \neq 0$, the boundary conditions on the Neumann directions are unmodified and we require $\pi R = R\pi = \pi$, i.e.

$$\pi = \begin{pmatrix} \delta^m{}_n & \\ & \end{pmatrix}, \qquad R = \begin{pmatrix} \delta^m{}_n & \\ & R^i{}_j \end{pmatrix}. \tag{4.2}$$

Then, the projector on the Dirichlet directions is defined by $Q = 1 - \pi$.

One can check that $\tau_+ = \rho^{\tilde{\beta}}_M$ is given by

$$\mathcal{R} = \begin{pmatrix} 1 & \tilde{\beta} \\ & 1 \end{pmatrix} \begin{pmatrix} r & \\ & -r^t \end{pmatrix} \begin{pmatrix} 1 & -\tilde{\beta} \\ & 1 \end{pmatrix} = \begin{pmatrix} r & -r\,\tilde{\beta} - \tilde{\beta}\,r^t \\ & -r^t \end{pmatrix}, \tag{4.3}$$

where $r = \pi - Q$. The last expression can be compared with the one in (2.1) to see that now $\mathcal{R}$ is an upper triangular matrix while before it was a lower triangular matrix. It was already suggested in [16] to relax the condition that $\mathcal{R}$ is a lower triangular matrix, but no explicit form was given. Here, the form (4.3) for the matrix $\mathcal{R}$ is a natural consequence of the definition of the generalized tangent bundle $\rho^{\tilde{\beta}}_M$. Using the gluing conditions $\Psi = \mathcal{R}\Psi$ and $\psi_- = R\psi_+$, one obtains

$$Q(G^{-1} - \tilde{\beta})Q^t + Q(G^{-1} + \tilde{\beta})Q^t R^t = 0\,. \tag{4.4}$$

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The explicit expression for the projector Q and the property $\tilde\beta|_{\mathrm{Ann}\,T^\omega_M}=0$ can be used to get

$$(R^{-1})^t = id_{T^*_M}\oplus(-id_{\mathrm{Ann}\,T^\omega_M})\oplus\big\{-(g^{-1}-\tilde\beta)^{-1}(g^{-1}+\tilde\beta)\big\} \tag{4.5}$$

acting on $T^*_N|_M\simeq T^*_M\oplus\mathrm{Ann}\,T^\omega_M\oplus E^*$. The metric g is now the restriction of the full metric G to E. Here, we are concerned with ordinary topological A-branes for which $I_+=I_-=I$. The boundary condition $j_++j_-=0$, where $j_\pm=\omega(\psi_\pm,\psi_\pm)$, gives $R^t\omega R=-\omega$. Using $\omega=GI$ and $R^tGR=G$, one finds that the matrix R anticommutes with the complex structure I of the target space manifold N. This means that R is of the form

$$R=\begin{pmatrix} & R_1\\ R_2 & \end{pmatrix} \tag{4.6}$$

with $R_1: T_N^{(0,1)}\to T_N^{(1,0)}$ and $R_2: T_N^{(1,0)}\to T_N^{(0,1)}$.

The stability condition for topological branes can be derived by requiring that the spectral flow operators

$$S_\pm=\Omega(\psi_\pm,\dots,\psi_\pm)=\frac{1}{n!}\Omega_{i_1\cdots i_n}\psi^{i_1}_\pm\cdots\psi^{i_n}_\pm \tag{4.7}$$

match properly on the boundary of the Riemann surface.[5] Here, N is a Calabi-Yau manifold and Ω is the holomorphic $(n,0)$-form. For topological A-branes the stability condition is

$$S_+=e^{i\alpha}\bar S_- \tag{4.8}$$

with α a real constant. Using the gluing condition $\psi_+=R_2^{-1}\psi_-$, the last equation yields

$$\Omega_{1\cdots n}\det(R_2^{-1})=e^{i\alpha}\bar\Omega_{1\cdots n}\,. \tag{4.9}$$

The equation (4.5) gives

$$\det(R_2^{-1})=(-1)^{p/2}(-1)^{2k}\det\big\{(G^{-1}-\tilde\beta)^{-1}(G^{-1}+\tilde\beta)|_{E^{*(1,0)}}\big\}\,, \tag{4.10}$$

where we have used that $(R_2^{-1})^t: T_N^{*(1,0)}\to T_N^{*(0,1)}$ and that $\dim|_{\mathbb R}(\mathrm{Ann}\,T^\omega_M)_x=\dim|_{\mathbb R}(T^*_M)_x=p$ with $p=n-2k$. Let us define the following matrices

$$G|_E=\begin{pmatrix} & g\\ g^t & \end{pmatrix},\qquad \tilde\beta=\begin{pmatrix}\tilde\beta_{(2,0)} & \tilde\beta_{(1,1)}\\ -\tilde\beta^t_{(1,1)} & \tilde\beta_{(0,2)}\end{pmatrix}, \tag{4.11}$$

where now $g: E^{(0,1)}\to E^{*(1,0)}$ and $\tilde\beta: E^*\to E$. These matrices are used to rewrite the matrix in equation (4.10) in terms of (p,q)-type components of $\tilde\beta$. To compute the determinant in (4.10) we need the following identities

$$\begin{aligned} g\tilde\beta_{(0,2)} &= -(1-g\tilde\beta^t_{(1,1)})(g^t\tilde\beta_{(2,0)})^{-1}(1-g^t\tilde\beta_{(1,1)})\,,\\ g^t\tilde\beta_{(2,0)} &= -(1-g^t\tilde\beta_{(1,1)})(g\tilde\beta_{(0,2)})^{-1}(1-g\tilde\beta^t_{(1,1)})\,, \end{aligned} \tag{4.12}$$

[5] We use the conventions of [19] where the BRST operator of the topological A-model is $Q_A=Q_++\bar Q_-$ and ψ^i_+ and $\psi^{\bar i}_-$ are the scalar fields on the Riemann surface.

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which can be derived by the fact that $\omega|_E\tilde{\beta}$ is an almost complex structure on E^*, i.e. $(\omega|_E\tilde{\beta})^2 = -1$. Then, one obtains

$$(G^{-1} - \tilde{\beta})^{-1}(G^{-1} + \tilde{\beta})|_{E^*(1,0)} = (1 - g^t\tilde{\beta}_{(1,1)})^{-1}g^t\tilde{\beta}_{(2,0)}\,, \tag{4.13}$$

whose determinant is easy to compute. Multiplying the equation (4.9) by $\Omega_{1\cdots n}$ and using the proportionality relation $|\Omega_{1\cdots n}|^2 \propto \sqrt{\det G}$, it follows that

$$(\Omega_{1\cdots n})^2 \det\tilde{\beta}_{(2,0)} \propto e^{i\alpha}\sqrt{\det G}\det\left\{(g^t)^{-1} - \tilde{\beta}_{(1,1)}\right\}. \tag{4.14}$$

The identities (4.12) can be used to get

$$\frac{\det\left(G^{-1} + \tilde{\beta}\right)}{\det G^{-1}} = 2^{2k}\det g\det\left\{(g^t)^{-1} - \tilde{\beta}_{(1,1)}\right\} \tag{4.15}$$

by which the equation (4.14) is rewritten as

$$(\Omega_{1\cdots n})^2 \det\tilde{\beta}_{(2,0)} \propto e^{i\alpha}\frac{\sqrt{\det G}}{\sqrt{\det G|_E}}\frac{\det\left(G^{-1} + \tilde{\beta}\right)}{\det G^{-1}}\,. \tag{4.16}$$

A lagrangian A-brane is stable if it is a special lagrangian submanifold M of the Calabi-Yau manifold N. This means that M is a calibrated submanifold with respect to the holomorphic $(n,0)$-form on N or in other words that M is defined using a closed form on N. Following this idea, we want to find a form on N by which we can define our isotropic submanifolds to be stable. Then, one can rewrite this form in terms of the geometric data found before.

Recall that $T^*_N|_M \simeq \operatorname{Ann}T^\omega_M \oplus T^*_M \oplus E^*$. Take $i = 1,\ldots,p$ and choose a complex basis $\{f^i;\bar{f}^i\}$ for $\operatorname{Ann}T^\omega_M \oplus T^*_M$ such that $\{\operatorname{Im} f^i\}$ and $\{\operatorname{Re} f^i\}$ span $\operatorname{Ann}T^\omega_M$ and T^*_M respectively. The complex basis $\{f^{p+1},\ldots,f^n;\bar{f}^{p+1},\ldots,\bar{f}^n\}$ can be chosen to span E^* so that $\{f^1,\ldots,f^n;\bar{f}^1,\ldots,\bar{f}^n\}$ is a complex basis for $T^*_N|_M$. In terms of this basis

$$\Omega|_{T^\omega_M} = \Omega_{1\cdots n}\operatorname{Re} f^1\wedge\cdots\wedge\operatorname{Re} f^p\wedge f^{p+1}\wedge\cdots\wedge f^n\,. \tag{4.17}$$

The last (n,0)-form can be multiplied by the $2k$-form $(\tilde{\beta}^{-1})^k$ to provide the volume form $\mathrm{vol}_{T^\omega_M}$ on T^ω_M. The property $(\omega|_E\tilde{\beta})^2 = -1$ gives

$$\tilde{\beta}^{-1} = \omega|_E\tilde{\beta}\omega^t|_E = \begin{pmatrix} -g\tilde{\beta}_{(0,2)}g^t & -g\tilde{\beta}^t_{(1,1)}g \\ g^t\tilde{\beta}_{(1,1)}g^t & -g^t\tilde{\beta}_{(2,0)}g \end{pmatrix}, \tag{4.18}$$

which can also be obtained computing the inverse of the matrix $\tilde{\beta}$ in (4.11) and using the identities (4.12). We will need the definition of the pfaffian of a skew-symmetric $2n\times 2n$ matrix, which we recall here. Consider a $2n$-dimensional riemannian manifold and a two-form $\alpha = A_{ab}e^*_a\wedge e^*_b$ associated to a skew-symmetric $2n\times 2n$ matrix A with elements A_{ab}. The pfaffian of A is defined by the equation

$$\frac{1}{n!}\alpha^n = \mathrm{Pf}(A)e^*_1\wedge\cdots\wedge e^*_{2n}\,. \tag{4.19}$$

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Using this definition, we find

$$
\begin{aligned}
\frac{1}{k!}\Omega|_{T^\omega_M}\wedge(\tilde\beta^{-1})^k &= \Omega_{1\cdots n}\mathrm{Pf}\big\{(\tilde\beta^{-1})_{(0,2)}\big\}\,\mathrm{Re}\,f^1\wedge\cdots\wedge\mathrm{Re}\,f^p\wedge f^{p+1}\wedge\cdots\wedge f^n\wedge\\
&\qquad\qquad\qquad\qquad\wedge\bar f^{p+1}\wedge\cdots\wedge\bar f^n\\
&=\Omega_{1\cdots n}\frac{\mathrm{Pf}\big\{(\tilde\beta^{-1})_{(0,2)}\big\}}{\sqrt{\det G|_{T^\omega_M}}}\mathrm{vol}_{T^\omega_M}\,,
\end{aligned}
\tag{4.20}
$$

where $(\tilde\beta^{-1})_{(0,2)}$ is the $(0,2)$-type component of $\tilde\beta^{-1}$, i.e. $(\tilde\beta^{-1})_{(0,2)}=-g^t\tilde\beta_{(2,0)}g$.

In the last equation we have used the definition of a volume form given in [18]. If (N,G) is a riemannian manifold, an oriented tangent m-plane on N is a m-dimensional vector subspace V of $(T_N)_x$ equipped with an orientation. The volume form vol_V on V is a m-form on V, which is given by the combination of the metric $G|_V$ on V with the orientation on V. In our case $(T^\omega_M)_x$ $\forall x\in M$ is an oriented tangent $(4k+p)$-plane on N. Thus, the combination of $G|_{(T^\omega_M)_x}$ with the orientation on $(T^\omega_M)_x$ gives a $(4k+p)$-form on $(T^\omega_M)_x$, which is the volume form $\mathrm{vol}_{(T^\omega_M)_x}$ on $(T^\omega_M)_x$ for all $x\in M$. The form in (4.20) is a $(n+2k)$-form on T^ω_M, which is a $(4k+p)$-form because $n=p+2k$.

The properties $\mathrm{Pf}(BAB^t)=\det(B)\mathrm{Pf}(A)$ and $\big(\mathrm{Pf}(A)\big)^2=\det(A)$, where B is an arbitrary matrix of rank $2n$, are used to rewrite the equation (4.20) as

$$
\frac{1}{k!}\Omega|_{T^\omega_M}\wedge(\tilde\beta^{-1})^k\propto\Omega_{1\cdots n}\frac{\det g\sqrt{\det\tilde\beta_{(2,0)}}}{\sqrt{\det G|_{T^\omega_M}}}\mathrm{vol}_{T^\omega_M}\,. \tag{4.21}
$$

The condition (4.16) derived from the matching of the spectral flow operators on the boundary can be used for the last equation to give the stability condition in terms of a form on N. Using the identity $\det G|_E\det G=(\det G|_{T^\omega_M})^2$, the stability condition for isotropic A-branes is

$$
\frac{1}{k!}\Omega|_{T^\omega_M}\wedge(\tilde\beta^{-1})^k\propto e^{i\frac{\alpha}{2}}\sqrt{\frac{\det\big(G^{-1}+\tilde\beta\big)}{\det G^{-1}}}\mathrm{vol}_{T^\omega_M}\,. \tag{4.22}
$$

The stability condition just derived is in agreement with the condition for a lagrangian A-brane to be stable. It is known that special lagrangian submanifolds are stable lagrangian A-branes. When an isotropic A-brane is a lagrangian submanifold, $T^\omega_M=T_M$, $k=0$ and $\tilde\beta=0$. Therefore, the equation (4.22) gives

$$
\Omega|_{T_M}\propto e^{i\frac{\alpha}{2}}\mathrm{vol}_{T_M}\,, \tag{4.23}
$$

which is the condition for a lagrangian submanifold to be special.

Note that the study of stability for topological A-branes with a non-trivial field strength on the worldvolume of the brane was performed in [20] using the supersymmetric Born-Infeld action. The authors of [12] found agreement with these results using the worldsheet approach. However, the case of the five dimensional coisotropic A-brane in T^6 or $T^2\times K3$

was not analyzed in [20]. For the isotropic non-lagrangian A-branes studied in this paper the case is rather different because the brane is equipped with a non-trivial bi-vector living in the normal direction of the brane. For a Calabi-Yau three-fold which is T^6 or $T^2 \times K3$, in addition to three dimensional lagrangian A-branes and five dimensional coisotropic A-branes, there are one dimensional isotropic A-branes, whose stability condition is given by (4.22) with $k = 1$. For a Calabi-Yau four-fold N we want to mention the case of two dimensional isotropic A-branes, whose stability condition is still given by (4.22) with $k = 1$.

5. Conclusion and outlook

In this paper we have shown the existence of isotropic A-branes whenever they are homological non-trivial cycles. We have also given the stability condition for these cycles by the worldsheet approach. Using Poincaré duality, it was argued that coisotropic and isotropic A-branes are dual cycles but with different differential structures on them. The coisotropic A-branes are equipped with a non-trivial curvature for the connection of a line bundle on the worldvolume of the brane while the isotropic ones are equipped with a non-trivial bi-vector in the normal direction of the brane. In the paper [4] it was argued that if Kontsevich's conjecture is true, the Fukaya category should be enlarged with coisotropic A-branes. It is then natural to propose that isotropic A-branes give a further enlargement of the Fukaya category.

Kontsevich's conjecture is about the equivalence of two triangulated categories for mirror manifolds N and $\hat{N}$: the derived category of coherent sheaves $D^b(N)$ and the derived Fukaya category $DF(\hat{N})$. In physics terms $D^b(N)$ is the category of topological B-branes while $DF(N)$ is the category of lagrangian A-branes. For a lagrangian A-brane to belong to $DF(N)$ one has to require an anomaly cancellation condition. We recall that at the classical level the boundary conditions for topological A-branes have been defined to preserve the axial R-symmetry, which could be broken at the quantum level. It turns out that the condition for a lagrangian A-brane to be special corresponds to the vanishing of the Maslov class, which is responsible for the anomaly. In mathematical terms this means that the lagrangian submanifold is gradable in the sense of Kontsevich. The anomaly free condition for coisotropic A-branes was studied in [21] based on a proposal from the stability condition worked out in [12]. The stability condition for coisotropic A-branes dictates a generalization of the Maslov class, whose zero gives the anomaly cancellation and graded coisotropic A-branes. It is clear that an analogous study for isotropic A-branes should be done based on the stability condition worked out here. This would correspond to graded isotropic A-branes, which together with the graded coisotropic ones might give the full enlargement of the Fukaya category from a physics perspective.

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